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the case of the descriptive properties. In the case of a function of n -variables, we may project upon an $(n - 1)$ space, an $(n - 2)$ space, etc.

The proofs of the preceding theorems are contained in a paper that has been offered to the *Transactions of the American Mathematical Society*.

¹ *Ann. Math. Princeton*, **18**, 1917 (147).

² For the terminology cf. Denjoy, *J. Math., Paris*, ser. 7, **1**, 1915 (122-125).

³ Cf. for example, Fréchet, *Rend. Circ. Math. Palermo*, **22**, 1907, p. 1; and Hausdorff, *Grundzüge der Mengenlehre*, 1914, p. 211.

⁴ Hausdorff, l. c., p. 315.

⁵ For the case where only measurable sets are admitted cf., for example, de la Vallée Poussin, *Cours d'Analyse*, **2**, 1912, p. 114. For the linear case of general (not necessarily measurable) sets cf. Blumberg, *Bull. Amer. Math. Soc.*, **25**, 1919 (350).

⁶ Cf. Denjoy, *Bull. Soc. Math. France*, **43**, 1915 (165).

⁷ In this connection cf. Gâteaux, *Ibid.*, **47**, 1919 (47).

GENERALIZED LIMITS IN GENERAL ANALYSIS

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It is well known that to each of the various methods for summing divergent series there corresponds an analogous method for summing divergent integrals. It is readily seen that similar methods may be used for obtaining types of generalized derivatives of a function at a point where the ordinary derivative fails to exist. Likewise, in any other case in Analysis where we wish to associate a limit with a variable that oscillates, we will naturally be led to make use of methods that have been tried out in the case of divergent series.

It would be manifestly poor economy of time and thought to elaborate for each of these special theories such fundamental results as are common to them all, if these results can be obtained in one central theory that includes all the others. According to a principle of generalization formulated by E. H. Moore and stated by him on several occasions,¹ the existence of such a general theory is implied by the analogies found among the various special theories. It is natural to designate this general theory as the theory of generalized limits in General Analysis.

It is the purpose of the present communication to illustrate the nature and usefulness of this theory by outlining the proof of a theorem in it which is a generalization of one of the important theorems in the theory of divergent series. This latter theorem is the Knopp-Schnee-Ford theorem² with regard to the equivalence of the Cesàro and Hölder means of order k

for summing divergent series. Our genreal theorem includes as special cases this theorem, the analogous theorem of Landau³ for divergent integrals, and a further new theorem with regard to the equivalence of generalized derivatives of the Cesàro and Hölder type. It can also be extended to the case of multiple limits, so that it includes new theorems, analogous to those mentioned, with regard to multiple series, multiple integrals, and partial derivatives. As this latter extension involves more complicated formulas, it will not be dealt with in the present communication.

Following the terminology introduced by E. H. Moore,⁴ we indicate the basis of our general theory as follows:

$(\mathfrak{A}; \mathfrak{P}; \mathfrak{S}; \mathfrak{G}^{\text{on } \mathfrak{S} \text{ to } \mathfrak{A}}; \mathfrak{H}^{\text{on } \mathfrak{S} \text{ to } \mathfrak{A}}; \mathfrak{F}^{\text{on } \mathfrak{S} \text{ to } \mathfrak{A}}; \varphi_0; J^{\text{on } \mathfrak{G} \text{ to } \mathfrak{H}} \text{ on } \mathfrak{H} \text{ to } \mathfrak{F})$,

where $\mathfrak{A} \equiv [a]$ denotes the class of all real numbers a , $\mathfrak{P} \equiv [p]$ denotes a class of elements p , and $\mathfrak{S} \equiv [\sigma]$ denotes a class of sets σ of elements p of the range $\mathfrak{P}$; $\mathfrak{G} \equiv [\gamma]$, $\mathfrak{H} \equiv [\eta]$, and $\mathfrak{F} \equiv [\varphi]$ are three classes of functions γ , η , and φ , respectively, on $\mathfrak{S}$ to $\mathfrak{A}$ (we shall restrict ourselves throughout to the consideration of single-valued functions); φ_0 is a special function φ of the class $\mathfrak{F}$; and J is a function on $\mathfrak{G}$ to $\mathfrak{H}$ and on $\mathfrak{H}$ to $\mathfrak{F}$, that is, a functional transformation turning a function of the class $\mathfrak{G}$ into a function of the class $\mathfrak{H}$ or a function of the class $\mathfrak{H}$ into a function of the class $\mathfrak{F}$, denoted by $J\gamma$ or $J\eta$.

In order to make clear the relationship of our general theorem to the two special theorems referred to above, we will indicate here what the general basis reduces to in the particular instances III and IV:

$$\begin{aligned} \mathfrak{P}^{\text{III}} &\equiv [\text{all } n = 1, 2, \dots]; \quad \mathfrak{S} \equiv [\sigma_n \equiv (1, 2, \dots, n) | n]; \\ \mathfrak{G} &\equiv \mathfrak{H} \equiv \mathfrak{F} \equiv [\text{all } \gamma, \eta, \varphi \text{ on } \mathfrak{S} \text{ to } \mathfrak{A}]; \\ \varphi_0(\sigma_n) &= n(n); \quad (J\gamma)(\sigma_n) = \gamma(\sigma_1) + \gamma(\sigma_2) + \dots + \gamma(\sigma_n)(n). \end{aligned}$$

$$\begin{aligned} \mathfrak{P}^{\text{IV}} &\equiv [\text{all } a > 0]; \quad \mathfrak{S} \equiv [\sigma \equiv (\text{all } x, 0 < x < a) | (a > 0)]; \\ \mathfrak{G} &\equiv [\text{all functions that are finite and integrable} \\ &\quad (\text{Lebesgue}) \text{ on every finite interval } (0, a)]; \\ \mathfrak{H} &\equiv [\text{all } \eta = \int_0^x \gamma | x > 0]; \quad \mathfrak{F} \equiv [\text{all } \varphi = (\int_0^x \eta | x > 0)]; \\ \varphi_0(\sigma_a) &= a(a); \quad (J\gamma)(\sigma_a) = \int_0^a \gamma(\gamma a); \quad (J\eta)(\sigma_a) = \int_0^a \eta(\eta a). \end{aligned}$$

We next proceed to make certain postulates with regard to the nature of the elements in our basis, readily seen to be verified in the specific instances cited. Thus we require the class $\mathfrak{G}$ to have the linear property (L) as defined by E. H. Moore,⁴ and the property (P) defined by

$$(P) \quad \gamma_1 \cdot \gamma_2 \cdot \supset \cdot \gamma_1 \gamma_2 \mathfrak{G},$$

the arguments of the two γ 's being the same. It will then follow that $\gamma_P = \gamma_1 \cdot \gamma_2 \cdot \dots \cdot \gamma_n$ is of the class $\mathfrak{G}$. We require the class $\mathfrak{H}$ to have the properties (L) and (P), and the further property of being a sub-class of the

$\mathfrak{S}$, which property we shall denote by S_G . We require the class $\mathfrak{F}$ to have the properties (L) and (P), and the further property of being a sub-class of the class $\mathfrak{S}$, which property we shall designate as S_H . Hence is J also on $\mathfrak{F}$ to $\mathfrak{F}$.

We require the operation J to have the linear property (L) as defined by E. H. Moore, and the properties M_1 and M_2 , defined as follows:

$$\begin{aligned} (M_1) \quad a_1 < \gamma_1 < a_2 \cdot 0 \leq \gamma_2 \cdot \gamma_3 &= \gamma_1 \gamma_2 \cdot \sup \cdot a_1 (J\gamma_2) (\sigma) \leq (J\gamma_3) (\sigma) \leq a_2 (J\gamma_2) (\sigma) (\sigma), \\ (M_2) \quad a_1 < \gamma_1 < a_2 \cdot 0 \leq \gamma_2 \cdot \gamma_3 &= \gamma_1 \gamma_2 \cdot \sigma'' \sup \sigma' \cdot \sup \cdot a_1 [(J\gamma_2) (\sigma'') - (J\gamma_2) (\sigma')] \\ &\leq (J\gamma_3) (\sigma'') - (J\gamma_3) (\sigma') \\ &\leq a_2 [(J\gamma_2) (\sigma'') - (J\gamma_2) (\sigma')]. \end{aligned}$$

For the explanation of the other postulates we need certain auxiliary definitions. By the notation $\sigma' < \sigma''$, $\sigma'' > \sigma'$, we shall mean that σ'' contains all the elements of σ' and at least one element not found in σ' . For a given θ on $\mathfrak{S}$ to $\mathfrak{A}$, a given a , and a given σ' such that there exists $\sigma < \sigma'$, we shall write

$$\lim_{\sigma | \sigma < \sigma'} \theta(\sigma) = a$$

in the case that, corresponding to an arbitrary positive number e , there exists a $\sigma_e < \sigma'$ such that for every σ having the property $\sigma_e \subset \sigma < \sigma'$, $|\theta(\sigma) - a| < e$, or in symbols

$$e \sup :: \exists \sigma_e < \sigma', \sigma_e \subset \sigma < \sigma' \cdot \sup \cdot |\theta(\sigma) - a| < e.$$

This latter definition is based on the fundamental definition of limit in General Analysis given by E. H. Moore in the paper in these PROCEEDINGS referred to in the first foot-note.

We now postulate for the class $\mathfrak{S}$ the following properties:

$$\begin{aligned} (\mathbf{U})^5 \quad & \text{Either } \sigma' \cdot \sup \cdot \exists \mathbf{U}[\sigma < \sigma']^{\mathfrak{S}} \text{ or } \exists \sigma_0, \sigma' > \sigma_0 \cdot \sup \cdot \exists \mathbf{U}[\sigma < \sigma']^{\mathfrak{S}}, \\ (\mathbf{N})^5 \quad & \sigma' \cdot \sup \cdot \exists \mathbf{N}[\sigma > \sigma']. \end{aligned}$$

In the typical instances in view in the formation of this general theory, of the two alternatives in (U) one holds and the other does not hold; however it is not assumed that this disjunction between the two alternatives shall be presupposed. In order to avoid notational reference to these alternatives, it is convenient to introduce a property—of sets σ of $\mathfrak{S}$; if the first alternative holds, every σ of $\mathfrak{S}$ has the property—, in notation $\bar{\sigma}$; if the first alternative does not hold, the sets $\bar{\sigma}$ are the sets $\sigma > \sigma_0$; further for brevity a property $\cdot$ (the negation of $-$) is used; thus every σ is a $\bar{\sigma}$ or a $\dot{\sigma}$.

We define

$$\begin{aligned} \sigma'_{-1} &\equiv \mathbf{U}[\sigma < \sigma']^{\mathfrak{S}} (\bar{\sigma}), \\ \sigma'_1 &\equiv \mathbf{N}[\sigma > \sigma']^{\mathfrak{S}}, \quad \sigma'_{n+1} \equiv \mathbf{N}[\sigma > \sigma'_n] (\sigma | n = 1, 2, 3 \dots), \\ \bar{\varphi}(\sigma) &\equiv \varphi(\sigma_{-1}) (\bar{\sigma}), \quad \bar{\varphi}(\sigma) \equiv 0 (\dot{\sigma}), \end{aligned}$$

and with regard to the functions $\bar{\varphi}$ we postulate

(F) all functions $\bar{\varphi}$ are of the class $\mathfrak{F}$.

We also define the notation

$$(1) \quad (D\theta)(\sigma) = \alpha(\sigma)$$

with regard to every θ , α on $\mathfrak{S}$ to $\mathfrak{A}$, to mean that

$$\lim_{\sigma|\sigma < \sigma'} \frac{\theta(\sigma') - \theta(\sigma)}{\varphi_0(\sigma') - \varphi_0(\sigma)} = \alpha(\sigma')(\bar{\sigma}), \quad \theta(\sigma) = \alpha(\sigma)(\bar{\sigma}).$$

We further postulate as to the class $\mathfrak{S}$

$$(R) \quad \sigma' \cdot \sup \cdot \mathfrak{H}(\sigma'_1)_{-1} = \sigma'(\sigma'),$$

for the class $\mathfrak{S}$

$$(C) \quad \eta \cdot \sigma' \cdot \sup \cdot \mathfrak{H} \lim_{\sigma|\sigma < \sigma'} \eta(\sigma) = \bar{\eta}(\sigma')(\bar{\sigma}'),$$

and for the class $\mathfrak{F}$

$$(\Delta) \quad \mathfrak{H} D\varphi \equiv [(D\varphi)(\sigma) | \sigma] \mathfrak{S}.$$

For the operation J we postulate

$$(I_D) \quad \eta \cdot \sup \cdot (D(J\eta))(\sigma) = \eta(\sigma)(\sigma),$$

$$(I_J) \quad \varphi \cdot \sup \cdot (J(D\varphi))(\sigma) = \varphi(\sigma)(\sigma).$$

We wish finally to postulate certain properties of the special function φ_0 . We need first to introduce a further definition. We shall mean by the notation

$$\lim_{\sigma} \eta(\sigma) = \infty$$

that for every positive e there exists a σ_e (dependent on e) such that for every $\sigma > \sigma_e$, $\eta(\sigma) > e$, or in symbols

$$e : \sup \cdot \mathfrak{H} \sigma_e, \sigma > \sigma_e \cdot \sup \cdot \eta(\sigma) > e.$$

This definition also is based on E. H. Moore's definition of a limit referred to above. We also introduce for the sake of brevity the following notations

$$(2) \quad \varphi_{0n}(\sigma) = \varphi_0(\sigma) \cdot \varphi_0(\sigma_1) \cdot \varphi_0(\sigma_2) \cdot \dots \cdot \varphi_0(\sigma_{n-1}) \quad (n > 1), \quad \varphi_{01}(\sigma) = \varphi_0(\sigma),$$

$$(3) \quad \psi_n(\sigma_1) = \frac{1}{\varphi_{0,n-1}(\sigma_1)} \quad (n > 1).$$

We now postulate for φ_0 that

$$(I) \quad \varphi_0 \text{ is a real and positive function of } \sigma,$$

$$(II) \quad \lim_{\sigma|\sigma < \sigma_n} [\varphi_0(\sigma'_n) - \varphi_0(\sigma)] = 1 \quad (\sigma'_n),$$

$$(III) \quad \frac{1}{\varphi_{0n}(\sigma)} (J^n \eta)(\sigma) \text{ as function of } \sigma \text{ is of the class } \mathfrak{F}(n),$$

the symbol $J^n \eta$ indicating that the operation J has been repeated n times

$$(IV) \quad \varphi_0(\sigma_n) \text{ as a function of } \sigma \text{ is of the class } \mathfrak{F}(n),$$

$$(V) \quad \lim_{\sigma} \varphi_{0n}(\sigma) = \infty \quad (n)$$

$$(VI) \quad (D\varphi_{0n})(\sigma) = n\varphi_{0,n-1}(\sigma) > 0 \quad (\sigma, n > 1), \quad (D\varphi_{01})(\sigma) = 1 \quad (\sigma),$$

$$(VII) \quad (D\psi_n)(\sigma_1) = -\frac{n-1}{\varphi_{0n}(\sigma)} \quad (\sigma, n > 1).$$

We have then as the foundation of our theory:

$$\Sigma \equiv (\mathfrak{A}; \mathfrak{B}; \mathfrak{C}; \mathfrak{G}^{\text{on}} \mathfrak{C} \text{ to } \mathfrak{A}, LP; \mathfrak{S}^{\text{on}} \mathfrak{C} \text{ to } \mathfrak{A}, LPS_G C;$$

$$\mathfrak{F}^{\text{on}} \mathfrak{C} \text{ to } \mathfrak{A}, LPS_H \Delta; \varphi_0 \mathfrak{F}. \text{ I II III IV V VI VII.}$$

$$J^{\text{on}} \mathfrak{G} \text{ to } \mathfrak{S}. \text{ on } \mathfrak{S} \text{ to } \mathfrak{F}. LM_1 M_2 I_D I_J).$$

We can now readily prove the two following properties of the operations D and J as applied to the class $\mathfrak{F}$:

$$(4) \quad (D[\varphi_1 \varphi_2])(\sigma) = \varphi_2(\sigma) (D\varphi_1)(\sigma) + \bar{\varphi}_1(\sigma) (D\varphi_2)(\sigma) \quad (\sigma),$$

$$(5) \quad (J[\varphi_2(D\varphi_1)])(\sigma) = \varphi_1(\sigma) \varphi_2(\sigma) - (J[\bar{\varphi}_1(D\varphi_2)])(\sigma) \quad (\sigma).$$

We next formulate a further limit definition that is also based on the definition of E. H. Moore to which reference has already been made. We shall say that $\eta(\sigma)$ approaches a limit a as σ if, corresponding to every positive ϵ , we can find a σ_ϵ such that for $\sigma > \sigma_\epsilon$ we have $|\eta(\sigma) - a| < \epsilon$, or in symbols

$$(6) \quad \lim_{\sigma} \eta(\sigma) = a :: \epsilon : \exists \sigma_\epsilon, \sigma > \sigma_\epsilon \Rightarrow |\eta(\sigma) - a| < \epsilon.$$

We are now ready to define the two generalized limits with which we are concerned. Given any function $\eta(\sigma)$, we set

$$(7) \quad (C_n \eta)(\sigma) = [n!/\varphi_{0n}(\sigma)](J^n \eta)(\sigma) \equiv \frac{n!}{\varphi_0(\sigma) \varphi_{0(1)} \cdots \varphi_{0(n-1)}} (J^n \eta)(\sigma),$$

$$(8) \quad (M\eta)(\sigma) = [1/\varphi_0(\sigma)](J\eta)(\sigma),$$

$$(9) \quad (H_n \eta)(\sigma) = (M^n \eta)(\sigma),$$

where C and H are used, as is customary, to connote Cesàro and Hölder. If for a fixed n $(C_n \eta)(\sigma)$ exists, we define this limit as the generalized limit of type (C_n) for $\eta(\sigma)$. If $\lim_{\sigma} (H_n \eta)(\sigma)$ exists, we define this limit as the generalized limit of type (H_n) for $\eta(\sigma)$.

We can now prove the equivalence of these two generalized limits by a method similar to that devised by Schur⁶ for proving the special theorem in the case of series. The proof breaks up into three lemmas and a theorem.

LEMMA 1. *If we represent by E the identical functional operation, $E\theta = \theta(\theta)$, we have the identity*

$$(10) \quad \left(\left(\frac{n-1}{n}M + \frac{1}{n}E\right)(C_n\eta)\right)(\sigma) = (M(C_{n-1}\eta))(\sigma).$$

LEMMA 2. $\mathfrak{H} \lim_{\sigma} \varphi(\sigma) = a \cdot |\varphi(\sigma)| < a_1(\sigma) \cdot \mathfrak{D}.$

$$\mathfrak{H} \lim_{\sigma} [\varphi_{0n}(\sigma)]^{-1} (J\varphi_n)(\sigma) = \frac{a}{n},$$

where

$\varphi_n(\sigma) = \varphi_{0,n-1}(\sigma)\varphi(\sigma) \ (n \geq 2), \quad \varphi_1(\sigma) = \varphi(\sigma),$
 $\varphi_{0n}(\sigma)$ being defined by (2).

LEMMA 3. *If we set*

$$\omega(\sigma) = \frac{1}{n} \varphi(\sigma) + \frac{n-1}{n} \cdot \frac{1}{\varphi_0(\sigma)} (J\varphi)(\sigma),$$

$$\mathfrak{H} \lim_{\sigma} \omega(\sigma) = a \cdot \mathfrak{D} \cdot \mathfrak{H} \lim_{\sigma} \varphi(\sigma) = a.$$

Theorem. $\mathfrak{H} \lim_{\sigma} (C_n\eta)(\sigma) = a \cdot \sim \cdot \mathfrak{H} \lim_{\sigma} (H_n\eta)(\sigma) = a.$

¹ E. H. Moore, *Introduction to a Form of General Analysis*, The New Haven Mathematical Colloquium, Yale Univ. Press, 1910, p. 3; "On a Form of General Analysis with Applications to Linear Differential and Integral Equations," *Atti IV Cong. Inter. Mat.* (Roma, 1908), **2**, p. 98; "On the Foundations of the Theory of Linear Integral Equations," *Bull. Amer. Math. Soc.*, Ser. 2, **18**, p. 339; "Definition of Limit in General Integral Analysis," these PROCEEDINGS, **1**, 1915, p. 628.

² K. Knopp, "Grenzwerte von Reihen bei der Annäherung an die Konvergenzgrenze," *Inauguraldissertation* (Berlin, 1907); W. Schnee, "Die Identität des Cesàroschen und Hölderschen Grenzwertes," *Math. Annalen*, **67**, 1909, pp. 110-125; W. B. Ford, "On the Relation between the Sum-Formulas of Hölder and Cesàro," *Amer. J. Math.*, **32**, 1910, pp. 315-326.

³ E. Landau, "Die Identität des Cesàroschen und Hölderschen Grenzwertes für Integrale," *Leipzig Berichte*, **65**, 1913, pp. 131-138.

⁴ Cf. the first three references in the first foot-note; also E. H. Moore, "On the Fundamental Functional Operation of a General Theory of Linear Integral Equations," *Proc. Fifth Inter. Congr. Math.* (Cambridge, 1912), **1**, 1913, pp. 230-255.

⁵ For the definition of these symbols see page 150 of the monograph referred to in the first footnote.

⁶ I. Schur, "Über die Äquivalenz der Cesàroschen und Hölderschen Mittelwerte," *Math. Annalen*, **74**, 1913, pp. 447-458.